

Comparing Estimators of Scale and Reliability Function of Frechet two Parameters Distribution

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Abstract

This paper deals with comparing different estimator of scale parameter (θ) and reliability function [R(t)] of the two parameters Frechet distribution considering the shape parameter (λ) is known, using different methods, these methods are least square, L – moment, proposed Bayes estimator and maximum likelihood method. The comparison has been done through simulation using different sample size ($n = 30,60,90,120,150$), and different set of initial values ($\lambda = 2,3,4, \theta = 0.5,1,1.5, c = 1,2$), each experiment replicated ($R = 500$), the comparison has been done using mean square error, all results explained in tables.

Keywords: Frechet two parameters, Scale parameter, Shape parameter, MSE, r^{th} Moment.

1. Introduction

The probability distribution of Gumbel and Frechet are widely used for frequency analysis of recorded meteorological data such as rainfall, temperature, wind speed, time to failure of certain produced items. Landwehr et al. (1979) discussed MLM to obtain quality estimator in sample, also Phien (1987) studied that MLM is the most efficient method since it have smallest possible variance, Gautam (2003) and Vivekanandan (2012) estimates the Frechet Distribution This distribution is a special case of the general distribution (extreme value distribution), it has been studied by Maurice Frechet (1927), also Fisher and Tippett (1928), and Gumbel (1958). Applications of the Frechet distribution in various fields given in Harlow (2002) showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Nadarajah and Kotz (2008) discussed the sociological models based on Frechet random variables. Further, Zaharim et al. (2009) applied Frechet distribution for analyzing the wind speed data. Mubarak (2011) studied the Frechet progressive type-II censored data with binomial removals.

The probability distribution function (*p.d.f*) of the random variable (x), which follow two parameters Frechet distribution (θ, λ) which are the scale and shape parameters as;

$$f(x; \theta, \lambda) = \lambda \theta^\lambda x^{-(\lambda+1)} e^{-\left(\frac{\theta}{x}\right)^\lambda} \quad x, \theta, \lambda > 0 \quad (1)$$

The cumulative probability function is;

$$F(x) = pr(X \leq x) = \int_0^x f(u)du = e^{-\left(\frac{\theta}{x}\right)^\lambda} \quad x, \theta, \lambda > 0 \quad (2)$$

the reliability function is;

$$R(x) = 1 - e^{-\left(\frac{\theta}{x}\right)^\lambda} \quad (3)$$

The formula of r^{th} moments about origin is;

$$\mu'_r = E(x^r) = \int_0^\infty x^r f(x; \theta, \lambda)dx = \theta^r \Gamma\left(\frac{\lambda-r}{\lambda}\right) \quad (4)$$

Then, the moment estimator are obtained from (4) as;

$$\bar{x} = \hat{\theta}_{mom} \Gamma\left(\frac{\lambda-1}{\lambda}\right) \quad (5)$$

$$\hat{\theta}_{mom} = \frac{\bar{x}}{\Gamma\left(\frac{\lambda-1}{\lambda}\right)} \quad (6)$$

2. Methods of estimation

2.1 Maximum Likelihood Method

The estimator obtained from maximizing the logarithm of likelihood function which is defined by;

$$L(x_1, x_2, \dots, x_n, \lambda, \theta) = \prod_{i=1}^n f(x_i; \lambda, \theta) = \lambda^n \theta^{n\lambda} \prod_{i=1}^n x_i^{-(\lambda+1)} e^{-\sum_{i=1}^n \left(\frac{\theta}{x_i}\right)^\lambda}$$

Then;

$$\ln L = n \ln \lambda + n\lambda \ln \theta - (\lambda + 1) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{\theta}{x_i}\right)^\lambda$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n\lambda}{\theta} - \lambda \theta^{\lambda-1} \sum_{i=1}^n x_i^{-\lambda}$$

$$put \quad \frac{\partial \log L}{\partial \theta} = 0$$

$$\hat{\theta}_{MLE} = \left[\frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad (7)$$

Which is an implicit function of (λ) Where (λ) is known parameter.

2.2 Least Square Estimator

This method is well known in statistics and mathematics, it depend on linear relation between two variables, one is called dependent variable (y_i), and second is independent variable (x_i), to apply it on two parameters Frechet, we apply the following;

Let;

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^\lambda}$$

Be *c. d. f* of two parameters Frechet, then;

$$\ln F(x) = -\left(\frac{\theta}{x}\right)^\lambda$$

$$-\ln F(x) = \left(\frac{\theta}{x}\right)^\lambda$$

$$\ln F^{-1}(x) = \left(\frac{\theta}{x}\right)^\lambda$$

$$\ln[\ln F^{-1}(x)] = \lambda(\ln \theta - \ln x_i)$$

$$y_i = \lambda \ln \theta - \lambda \ln x_i$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^n \ln \left[\ln \frac{1}{F(x_i)} \right] \quad \text{wher } F(x_i) = \frac{i}{n+1}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n \ln(x_i)$$

$$y_i = \ln \left[\ln \frac{1}{F(x_i)} \right] = \ln \left[\ln \left(\frac{n+1}{i} \right) \right]$$

We estimate (θ) by OLS as follows, and observation [$x_i = \ln(x_i)$], then;

$$\hat{\theta}_{OLS} = e^{\left(\bar{y} - \frac{\bar{x}}{\lambda}\right)} \tag{8}$$

2.3 Estimation using L - Moment

The estimation by this method depends on equating the formula of population moment (β_r) as;

$$\beta_r = \int_0^\infty x F^r(x) f(x) dx \tag{9}$$

With sample moment (b_r);

$$b_r = \frac{1}{n C_r^{n-1}} \sum_{i=1}^n C_r^{i-1} x_{(i)} \tag{10}$$

$$\beta_r = \int_0^\infty x e^{\left[-\left(\frac{\theta}{x}\right)^\lambda\right]^r} \lambda \theta^\lambda x^{-(\lambda+1)} e^{-\left(\frac{\theta}{x}\right)^\lambda} dx$$

$$\beta_r = \lambda \int_0^\infty \left(\frac{\theta}{x}\right)^\lambda e^{-\left(\frac{\theta}{x}\right)^{\lambda r}} e^{-\left(\frac{\theta}{x}\right)^\lambda} dx$$

Let;

$$\frac{\theta}{x} = y \rightarrow x = \frac{\theta}{y} \quad dx = -\frac{\theta}{y^2} dy$$

$$\begin{aligned} \beta_r &= \lambda \int_0^\infty y^\lambda e^{-ry^\lambda} e^{-y^\lambda} \frac{\theta}{y^2} dy \\ &= \theta \lambda \int_0^\infty y^{\lambda-2} e^{-(r+1)y^\lambda} dy \end{aligned}$$

Let;

$$z = (r + 1)y^\lambda \rightarrow \left(\frac{z}{r+1}\right)^{\frac{1}{\lambda}} = y$$

$$y = (z)^{\frac{1}{\lambda}} \left(\frac{1}{r+1}\right)^{\frac{1}{\lambda}} \quad dy = \frac{1}{\lambda} (z)^{\frac{1}{\lambda}-1} \left(\frac{1}{r+1}\right)^{\frac{1}{\lambda}} dz$$

$$\begin{aligned} \beta_r &= \theta \lambda \int_0^\infty \left[(z)^{\frac{1}{\lambda}}\right]^{\lambda-2} e^{-z} \frac{1}{\lambda} (z)^{\frac{1}{\lambda}-1} \left(\frac{1}{r+1}\right)^{\frac{1}{\lambda}} dz \\ &= \theta \left(\frac{1}{r+1}\right)^{\frac{1}{\lambda}} \int_0^\infty e^{-z} (z)^{1-\frac{2}{\lambda}+\frac{1}{\lambda}-1} dz \\ &= \frac{\theta}{(r+1)^{\frac{1}{\lambda}}} \int_0^\infty e^{-z} (z)^{-\frac{1}{\lambda}} dz \\ &= \frac{\theta}{(r+1)^{\frac{1}{\lambda}}} \Gamma\left(1 - \frac{1}{\lambda}\right) \end{aligned} \tag{11}$$

Then;

$$\beta_r = b_r \tag{12}$$

$$\beta_1 = \frac{\theta}{2^\lambda} \Gamma\left(1 - \frac{1}{\lambda}\right) \tag{13}$$

$$b_1 = \frac{1}{n(n-1)} \sum_{i=1}^n (i-1)x_{(i)}$$

Where $x_{(1)} \leq x_{(2)} \leq x_{(3)} \leq \dots \leq x_{(n)}$ are ordered observation of sample values x_1, x_2, \dots, x_n , from solve $[\beta_1 = b_1]$ and $[\beta_2 = b_2]$ we obtain the estimator by L – moment method, since here we have only one parameter, we apply equation (13).

2.4 Bayes Estimator

Bayes Estimator of scale parameter (θ), this Method consider (θ) as random variable have prior distribution $\pi(\theta)$ defined by:

$$\pi(\theta) = k\theta^c \quad \theta > 0$$

$$\text{Constant} \quad c, k > 0$$

According to this we find the posterior distribution

$$g(\theta|x) \propto L(x; \lambda, \theta)\pi(\theta)$$

$$\text{Now } f(x) = \int L(x; \lambda, \theta)\pi(\theta) d\theta$$

Then applying certain steps to find the posterior distribution, which is equal to:

$$g(\theta|x) = \frac{\lambda \theta^{c+n\lambda} T^{\frac{c+1}{\lambda}+n} e^{-\sum(\frac{\theta}{x_i})^\lambda}}{\Gamma(\frac{c+1}{\lambda} + n)}$$

Which is a p.d.f since

$$\int_{\forall \theta} g(\theta|x) d\theta = 1$$

Then according to squared error loss function

$$L = E(\hat{\theta} - \theta)^2$$

We find that the Bayes estimator for parameter θ is

$$\hat{\theta}_{Bayes} = \frac{\Gamma(\frac{c+2}{\lambda}+n)}{T^\lambda \Gamma(\frac{c+1}{\lambda}+n)}$$

$$\text{where } T = \sum_{i=1}^n \left(\frac{1}{x_i}\right)^\lambda$$

3. Simulation

The estimator using different sample size ($n = 30,60,90,120,150$), and the initial values for (λ) and (θ) are;

| | | | |
|-----------|-----|---|-----|
| λ | 2 | 3 | 4 |
| θ | 0.5 | 1 | 1.5 |
| C | 1 | 2 | |

The replicate of each experiment (R=500), the results ($\hat{\theta}$) then [$\hat{R}(t_i)$] are compared using statistical measure (MSE), and applying method of least square, L – moment, Moment, and Maximum Likelihood, all results explained in tables.

Estimation of Scale Parameters (θ) when (λ) is known as follows;

Table (1): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 2, \theta = 0.5, c=2$)

| n | | LS | L – Mom | Bayes | ML | Best |
|-----|-----------------------|----------|----------|----------|----------|-------|
| 30 | $\hat{\theta}$ | 0.376632 | 0.360632 | 0.510876 | 0.506671 | |
| | MSE($\hat{\theta}$) | 0.017145 | 0.051068 | 0.002325 | 0.002338 | Bayes |
| 60 | $\hat{\theta}$ | 0.370461 | 0.356793 | 0.505568 | 0.503474 | |
| | MSE($\hat{\theta}$) | 0.017736 | 0.030325 | 0.001095 | 0.001082 | ML |
| 90 | $\hat{\theta}$ | 0.368723 | 0.352914 | 0.503598 | 0.502205 | |
| | MSE($\hat{\theta}$) | 0.017922 | 0.026921 | 0.000764 | 0.000752 | ML |
| 120 | $\hat{\theta}$ | 0.368398 | 0.349229 | 0.502118 | 0.501075 | |
| | MSE($\hat{\theta}$) | 0.017757 | 0.025902 | 0.000545 | 0.000539 | ML |
| 150 | $\hat{\theta}$ | 0.365971 | 0.351353 | 0.502069 | 0.501234 | |
| | MSE($\hat{\theta}$) | 0.01843 | 0.025367 | 0.000448 | 0.000443 | ML |

Table (2): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 3, \theta = 0.5, c = 2$)

| n | | LS | L – Mom | Bayes | ML | Best |
|-----|-----------------------|----------|----------|----------|----------|-------|
| 30 | $\hat{\theta}$ | 0.467257 | 0.395318 | 0.504973 | 0.504037 | |
| | MSE($\hat{\theta}$) | 0.001216 | 0.013808 | 0.001057 | 0.001045 | ML |
| 60 | $\hat{\theta}$ | 0.459966 | 0.395585 | 0.502056 | 0.501591 | |
| | MSE($\hat{\theta}$) | 0.00168 | 0.012408 | 0.000509 | 0.000506 | ML |
| 90 | $\hat{\theta}$ | 0.456919 | 0.395337 | 0.501303 | 0.500994 | |
| | MSE($\hat{\theta}$) | 0.00192 | 0.012425 | 0.000325 | 0.000324 | ML |
| 120 | $\hat{\theta}$ | 0.454651 | 0.397026 | 0.500932 | 0.5007 | |
| | MSE($\hat{\theta}$) | 0.002095 | 0.011373 | 0.000241 | 0.00024 | ML |
| 150 | $\hat{\theta}$ | 0.453827 | 0.396122 | 0.501248 | 0.501062 | |
| | MSE($\hat{\theta}$) | 0.002161 | 0.011384 | 0.000188 | 0.000189 | Bayes |

Table (3): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 4, \theta = 0.5, c = 2$)

| n | | LS | L – Mom | Bayes | ML | Best |
|----|-----------------------|----------|----------|----------|----------|------|
| 30 | $\hat{\theta}$ | 0.501556 | 0.421859 | 0.503478 | 0.503473 | |
| | MSE($\hat{\theta}$) | 4.31E-05 | 0.007669 | 0.000563 | 0.000564 | Ls |
| 60 | $\hat{\theta}$ | 0.494008 | 0.420777 | 0.501695 | 0.501694 | |
| | MSE($\hat{\theta}$) | 5.15E-05 | 0.006937 | 0.000253 | 0.000253 | LS |

| | | | | | | |
|-----|---------------------|----------|----------|----------|----------|----|
| 90 | $\hat{\theta}$ | 0.490912 | 0.419503 | 0.500203 | 0.500203 | |
| | $MSE(\hat{\theta})$ | 9.29E-05 | 0.006905 | 0.000179 | 0.000179 | LS |
| 120 | $\hat{\theta}$ | 0.489019 | 0.420614 | 0.500444 | 0.500444 | |
| | $MSE(\hat{\theta})$ | 1.30E-04 | 0.006752 | 0.00012 | 0.00012 | LS |
| 150 | $\hat{\theta}$ | 0.365394 | 0.354234 | 0.501529 | 0.500695 | |
| | $MSE(\hat{\theta})$ | 1.86E-02 | 0.02458 | 0.000442 | 0.000439 | LS |

Table (4): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 2, \theta = 1, c=2$)

| n | | <i>LS</i> | <i>L – Mom</i> | <i>Bayes</i> | <i>ML</i> | <i>Best</i> |
|-----|---------------------|-----------|----------------|--------------|-----------|-------------|
| 30 | $\hat{\theta}$ | 0.247964 | 0.70249 | 1.020799 | 1.012397 | |
| | $MSE(\hat{\theta})$ | 5.68E-01 | 0.143085 | 0.008996 | 0.008577 | LS |
| 60 | $\hat{\theta}$ | 0.240603 | 0.71059 | 1.011521 | 1.007332 | |
| | $MSE(\hat{\theta})$ | 5.78E-01 | 0.118318 | 0.004451 | 0.004337 | ML |
| 90 | $\hat{\theta}$ | 0.238428 | 0.703446 | 1.006155 | 1.003372 | |
| | $MSE(\hat{\theta})$ | 5.81E-01 | 0.102626 | 0.002765 | 0.002724 | ML |
| 120 | $\hat{\theta}$ | 0.238121 | 0.701012 | 1.00558 | 1.003492 | |
| | $MSE(\hat{\theta})$ | 5.81E-01 | 0.100811 | 0.00204 | 0.002013 | ML |
| 150 | $\hat{\theta}$ | 0.236656 | 0.702827 | 1.004516 | 1.002846 | |
| | $MSE(\hat{\theta})$ | 5.83E-01 | 0.097626 | 0.001692 | 0.001674 | ML |

Table (5): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 3, \theta = 1, c = 2$)

| n | | <i>LS</i> | <i>L – Mom</i> | <i>Bayes</i> | <i>ML</i> | <i>Best</i> |
|-----|---------------------|-----------|----------------|--------------|-----------|-------------|
| 30 | $\hat{\theta}$ | 0.372851 | 0.792894 | 1.009851 | 1.00798 | |
| | $MSE(\hat{\theta})$ | 0.393674 | 0.055429 | 0.003972 | 0.003981 | Bayes |
| 60 | $\hat{\theta}$ | 0.367172 | 0.79479 | 1.002659 | 1.00173 | |
| | $MSE(\hat{\theta})$ | 0.40065 | 0.048435 | 0.001792 | 0.001783 | ML |
| 90 | $\hat{\theta}$ | 0.363727 | 0.79714 | 1.00343 | 1.002811 | |
| | $MSE(\hat{\theta})$ | 0.40498 | 0.045928 | 0.001329 | 0.001322 | ML |
| 120 | $\hat{\theta}$ | 0.363356 | 0.79346 | 1.001384 | 1.00092 | |
| | $MSE(\hat{\theta})$ | 0.405406 | 0.045636 | 0.001002 | 0.001012 | Bayes |
| 150 | $\hat{\theta}$ | 0.362001 | 0.793867 | 1.001816 | 1.001445 | |
| | $MSE(\hat{\theta})$ | 0.407114 | 0.044858 | 0.000751 | 0.000749 | ML |

Table (6): Estimation of Scale Parameters (θ) and Mean square error ($\lambda = 4, \theta = 1, c = 2$)

| n | | <i>LS</i> | <i>L – Mom</i> | <i>Bayes</i> | <i>ML</i> | <i>Best</i> |
|-----|---------------------|-----------|----------------|--------------|-----------|-------------|
| 30 | $\hat{\theta}$ | 0.430436 | 0.84139 | 1.005236 | 1.005227 | |
| | $MSE(\hat{\theta})$ | 3.25E-01 | 0.030507 | 0.002209 | 0.002219 | Bayes |
| 60 | $\hat{\theta}$ | 0.424277 | 0.83779 | 1.002349 | 1.002347 | |
| | $MSE(\hat{\theta})$ | 3.32E-01 | 0.029234 | 0.001068 | 0.001069 | Bayes |
| 90 | $\hat{\theta}$ | 0.421215 | 0.839739 | 1.001753 | 1.001752 | |
| | $MSE(\hat{\theta})$ | 3.35E-01 | 0.027605 | 0.000717 | 0.000716 | ML |
| 120 | $\hat{\theta}$ | 0.419678 | 0.839229 | 1.001182 | 1.001181 | |
| | $MSE(\hat{\theta})$ | 3.37E-01 | 0.027143 | 0.000498 | 0.000497 | ML |
| 150 | $\hat{\theta}$ | 0.418768 | 0.839136 | 1.00049 | 1.00049 | |
| | $MSE(\hat{\theta})$ | 3.38E-01 | 0.026966 | 0.000414 | 0.000413 | ML |

Table (7): Estimation of Scale Parameters (θ) and Mean square error ($\lambda = 2, \theta = 1.5, c = 1$)

| n | | <i>LS</i> | <i>L – Mom</i> | <i>Bayes</i> | <i>ML</i> | <i>Best</i> |
|-----|---------------------|-----------|----------------|--------------|-----------|-------------|
| 30 | $\hat{\theta}$ | 0.164262 | 1.086442 | 1.532439 | 1.526067 | |
| | $MSE(\hat{\theta})$ | 1.79E+00 | 1.079284 | 0.022238 | 0.02169 | ML |
| 60 | $\hat{\theta}$ | 0.156358 | 1.062928 | 1.510098 | 1.506956 | |
| | $MSE(\hat{\theta})$ | 1.81E+00 | 0.310369 | 0.010326 | 0.01023 | ML |
| 90 | $\hat{\theta}$ | 0.155192 | 1.052071 | 1.507966 | 1.505874 | |
| | $MSE(\hat{\theta})$ | 1.81E+00 | 0.230109 | 0.006125 | 0.00608 | ML |
| 120 | $\hat{\theta}$ | 0.154372 | 1.05465 | 1.50421 | 1.502644 | |
| | $MSE(\hat{\theta})$ | 1.81E+00 | 0.236126 | 0.004446 | 0.004426 | ML |
| 150 | $\hat{\theta}$ | 0.15389 | 1.051368 | 1.50366 | 1.502408 | |
| | $MSE(\hat{\theta})$ | 1.81E+00 | 0.23051 | 0.003965 | 0.003951 | ML |

Table (8): Estimation of Scale Parameters (θ) and mean square error ($\lambda = 4, \theta = 1.5, c = 1$)

| n | | <i>LS</i> | <i>L – Mom</i> | <i>Bayes</i> | <i>ML</i> | <i>Best</i> |
|-----|---------------------|-----------|----------------|--------------|-----------|-------------|
| 30 | $\hat{\theta}$ | 0.369938 | 1.260877 | 1.505293 | 1.506849 | |
| | $MSE(\hat{\theta})$ | 1.28E+00 | 0.069902 | 0.005293 | 0.005322 | Bayes |
| 60 | $\hat{\theta}$ | 0.36347 | 1.26317 | 1.502937 | 1.503717 | |
| | $MSE(\hat{\theta})$ | 1.29E+00 | 0.062522 | 0.002369 | 0.002377 | Bayes |
| 90 | $\hat{\theta}$ | 0.361354 | 1.261728 | 1.502491 | 1.503011 | |
| | $MSE(\hat{\theta})$ | 1.30E+00 | 0.060659 | 0.001546 | 0.00155 | Bayes |
| 120 | $\hat{\theta}$ | 0.359962 | 1.260681 | 1.50211 | 1.502501 | |
| | $MSE(\hat{\theta})$ | 1.30E+00 | 0.060208 | 0.001142 | 0.001144 | Bayes |
| 150 | $\hat{\theta}$ | 0.358939 | 1.260793 | 1.502814 | 1.503127 | |
| | $MSE(\hat{\theta})$ | 1.30E+00 | 0.059568 | 0.000908 | 0.00091 | Bayes |

Estimators of Reliability function as follows;

Table (9): Reliability estimator when $(\lambda = 2, \theta = 0.5, c = 2)$

| n | t | Rea | \hat{R}_{LS} | $MSE(\hat{R}_t)$ | \hat{R}_{LM} | $MSE(\hat{R})$ | \hat{R}_{Bayes} | $MSE(\hat{R}_B)$ | \hat{R}_{MLE} | $MSE(\hat{R})$ | Be |
|-----|-----|----------|----------------|------------------|----------------|----------------|-------------------|------------------|-----------------|----------------|------|
| 30 | 0.5 | 0.632121 | 0.433465 | 0.044113 | 0.38947 | 0.081289 | 0.64548 | 4.89E-03 | 0.639514 | 4.77E-03 | MLE |
| | 1 | 0.221199 | 0.134762 | 0.008094 | 0.121002 | 0.016021 | 0.228296 | 0.001458 | 0.225025 | 0.001389 | MLE |
| | 1.5 | 0.105161 | 0.061838 | 0.002012 | 0.060309 | 0.006873 | 0.10988 | 0.000385 | 0.108184 | 0.000362 | MLE |
| | 2 | 0.060587 | 0.035699 | 0.000658 | 0.032752 | 0.001567 | 0.063968 | 1.44E-04 | 0.062954 | 1.34E-04 | MLE |
| | 2.5 | 0.039211 | 0.022731 | 0.000292 | 0.023775 | 0.002133 | 0.041226 | 6.37E-05 | 0.040564 | 5.96E-05 | MLE |
| 60 | 0.5 | 0.632121 | 0.423343 | 0.046371 | 0.390445 | 0.073727 | 0.637372 | 2.30E-03 | 0.634346 | 1.40E-03 | MLE |
| | 1 | 0.221199 | 0.129407 | 0.008744 | 0.120181 | 0.013913 | 0.225034 | 0.000677 | 0.223403 | 0.000659 | MLE |
| | 1.5 | 0.105161 | 0.060055 | 0.002112 | 0.055267 | 0.003354 | 0.107294 | 0.000178 | 0.106457 | 0.000174 | MLE |
| | 2 | 0.060587 | 0.034 | 0.000729 | 0.032033 | 0.001104 | 0.062178 | 6.50E-05 | 0.061681 | 6.27E-05 | MLE |
| | 2.5 | 0.039211 | 0.021863 | 0.000312 | 0.02148 | 0.000676 | 0.040225 | 2.78E-05 | 0.039899 | 2.68E-05 | MLE |
| 90 | 0.5 | 0.632121 | 0.421358 | 0.04616 | 0.386915 | 0.07054 | 0.634364 | 1.52E-03 | 0.632338 | 1.52E-03 | MLE |
| | 1 | 0.221199 | 0.127854 | 0.008938 | 0.116602 | 0.012699 | 0.224292 | 0.000424 | 0.223204 | 0.000416 | MLE |
| | 1.5 | 0.105161 | 0.058891 | 0.002203 | 0.055772 | 0.003793 | 0.106591 | 0.000104 | 0.106035 | 0.000102 | MLE |
| | 2 | 0.060587 | 0.033674 | 0.000741 | 0.030949 | 0.001058 | 0.061314 | 3.81E-05 | 0.060986 | 3.73E-05 | MLE |
| | 2.5 | 0.039211 | 0.021622 | 0.000317 | 0.020489 | 0.000479 | 0.039871 | 1.87E-05 | 0.039656 | 1.83E-05 | MLE |
| 120 | 0.5 | 0.632121 | 0.415776 | 0.048573 | 0.393586 | 0.066564 | 0.633933 | 1.24E-03 | 0.632411 | 1.23E-03 | MLE |
| | 1 | 0.221199 | 0.126513 | 0.00916 | 0.118701 | 0.012672 | 0.223981 | 0.000337 | 0.223165 | 0.000331 | MLE |
| | 1.5 | 0.105161 | 0.057808 | 0.002289 | 0.056272 | 0.003342 | 0.106298 | 8.74E-05 | 0.105882 | 8.60E-05 | MLE |
| | 2 | 0.060587 | 0.03311 | 0.000769 | 0.031789 | 0.001018 | 0.061388 | 3.05E-05 | 0.061141 | 3.00E-05 | MLE |
| | 2.5 | 0.039211 | 0.021547 | 0.000317 | 0.019789 | 0.00044 | 0.039613 | 1.32E-05 | 0.039452 | 1.29E-05 | MLE |
| 150 | 0.5 | 0.632121 | 0.414541 | 0.048847 | 0.39015 | 0.066604 | 0.633021 | 9.06E-04 | 0.631801 | 9.05E-04 | MLE |
| | 1 | 0.221199 | 0.125987 | 0.009219 | 0.118765 | 0.012293 | 0.22327 | 0.000263 | 0.222618 | 0.00026 | MLE |
| | 1.5 | 0.105161 | 0.057984 | 0.00226 | 0.0545 | 0.002896 | 0.106496 | 6.80E-05 | 0.106162 | 6.68E-05 | MLE |
| | 2 | 0.060587 | 0.032958 | 0.000776 | 0.031709 | 0.001076 | 0.060961 | 2.41E-05 | 0.060765 | 2.38E-05 | MLE |
| | 2.5 | 0.039211 | 0.021272 | 0.000327 | 0.020036 | 0.000434 | 0.039592 | 1.08E-05 | 0.039463 | 1.07E-05 | MLE |

Table (10): Reliability estimator when $(\lambda = 3, \theta = 0.5, c = 2)$

| n | t_i | <i>Real</i> | \widehat{R}_{LS} | $MSE(\widehat{R}_{LS})$ | \widehat{R}_{LM} | $MSE(\widehat{R}_{LM})$ | \widehat{R}_{Bayes} | $MSE(\widehat{R}_{Bayes})$ | \widehat{R}_{Mle} | $MSE(\widehat{R}_{Mle})$ | <i>Best</i> |
|-----|-------|-------------|--------------------|-------------------------|--------------------|-------------------------|-----------------------|----------------------------|---------------------|--------------------------|-------------|
| 30 | 0.5 | 0.632121 | 0.556308 | 0.00655 | 0.39854 | 0.068923 | 0.640333 | 4.53E-03 | 0.638321 | 4.50E-03 | MLE |
| | 1 | 0.117503 | 0.097101 | 0.000468 | 0.064512 | 0.004356 | 0.120769 | 0.000444 | 0.120142 | 0.000436 | MLE |
| | 1.5 | 0.03636 | 0.029757 | 4.91E-05 | 0.01965 | 0.000417 | 0.03772 | 5.09E-05 | 0.037515 | 4.98E-05 | MLE |
| | 2 | 0.015504 | 0.012693 | 8.79E-06 | 0.008152 | 7.09E-05 | 0.016159 | 9.75E-06 | 0.01607 | 9.54E-06 | MLE |
| | 2.5 | 0.007968 | 0.006523 | 2.32E-06 | 0.004176 | 1.80E-05 | 0.008244 | 2.54E-06 | 0.008199 | 2.49E-06 | MLE |
| 60 | 0.5 | 0.632121 | 0.539482 | 0.009006 | 0.395915 | 0.062911 | 0.636553 | 2.36E-03 | 0.635539 | 2.35E-03 | MLE |
| | 1 | 0.117503 | 0.092549 | 0.000648 | 0.0625 | 0.003512 | 0.119236 | 0.000227 | 0.118926 | 0.000225 | MLE |
| | 1.5 | 0.03636 | 0.028306 | 6.74E-05 | 0.018974 | 0.000346 | 0.037236 | 2.52E-05 | 0.037134 | 2.49E-05 | MLE |
| | 2 | 0.015504 | 0.012071 | 1.22E-05 | 0.008014 | 6.35E-05 | 0.015804 | 4.46E-06 | 0.01576 | 4.41E-06 | MLE |
| | 2.5 | 0.007968 | 0.006215 | 3.19E-06 | 0.004072 | 1.70E-05 | 0.008114 | 1.23E-06 | 0.008092 | 1.21E-06 | MLE |
| 90 | 0.5 | 0.632121 | 0.533205 | 0.010074 | 0.3914 | 0.06303 | 0.634694 | 1.59E-03 | 0.634016 | 1.59E-03 | MLE |
| | 1 | 0.117503 | 0.09073 | 0.000732 | 0.061237 | 0.003352 | 0.118535 | 0.000135 | 0.118329 | 0.000134 | MLE |
| | 1.5 | 0.03636 | 0.02782 | 7.46E-05 | 0.018659 | 0.000338 | 0.036976 | 1.58E-05 | 0.036909 | 1.57E-05 | MLE |
| | 2 | 0.015504 | 0.011852 | 1.36E-05 | 0.007837 | 6.25E-05 | 0.015631 | 2.70E-06 | 0.015602 | 2.69E-06 | MLE |
| | 2.5 | 0.007968 | 0.006092 | 3.61E-06 | 0.004081 | 2.35E-05 | 0.008061 | 7.70E-07 | 0.008046 | 7.65E-07 | MLE |
| 120 | 0.5 | 0.632121 | 0.528441 | 0.010969 | 0.395493 | 0.059937 | 0.635445 | 1.14E-03 | 0.634936 | 1.13E-03 | MLE |
| | 1 | 0.117503 | 0.089844 | 0.000779 | 0.061062 | 0.003352 | 0.118624 | 0.0001 | 0.11847 | 9.95E-05 | MLE |
| | 1.5 | 0.03636 | 0.027536 | 7.91E-05 | 0.018365 | 0.000341 | 0.036623 | 1.08E-05 | 0.036573 | 1.07E-05 | MLE |
| | 2 | 0.015504 | 0.011673 | 1.49E-05 | 0.007932 | 6.04E-05 | 0.015749 | 2.39E-06 | 0.015727 | 2.37E-06 | MLE |
| | 2.5 | 0.007968 | 0.006 | 3.93E-06 | 0.004057 | 1.61E-05 | 0.008035 | 5.58E-07 | 0.008024 | 5.55E-07 | MLE |
| 150 | 0.5 | 0.632121 | 0.525701 | 0.011495 | 0.396298 | 0.059086 | 0.635978 | 9.39E-04 | 0.63557 | 9.36E-04 | MLE |
| | 1 | 0.117503 | 0.089155 | 0.000814 | 0.061764 | 0.003291 | 0.117922 | 8.56E-05 | 0.1178 | 8.53E-05 | MLE |
| | 1.5 | 0.03636 | 0.027289 | 8.32E-05 | 0.01853 | 0.000331 | 0.036711 | 9.16E-06 | 0.036671 | 9.11E-06 | MLE |
| | 2 | 0.015504 | 0.011605 | 1.54E-05 | 0.007916 | 6.13E-05 | 0.015666 | 1.64E-06 | 0.015649 | 1.63E-06 | MLE |
| | 2.5 | 0.007968 | 0.005967 | 4.05E-06 | 0.004016 | 1.62E-05 | 0.008045 | 4.39E-07 | 0.008036 | 4.37E-07 | MLE |

Table (11): Reliability estimator when ($\lambda = 4, \theta = 0.5, c = 2$)

| n | t_i | <i>Real</i> | \widehat{R}_{LS} | $MSE(\widehat{R}_{LS})$ | \widehat{R}_{LM} | $MSE(\widehat{R}_{LM})$ | \widehat{R}_{Bayes} | $MSE(\widehat{R}_{Bayes})$ | \widehat{R}_{Mle} | $MSE(\widehat{R}_{Mle})$ | <i>Best</i> |
|-----|-------|-------------|--------------------|-------------------------|--------------------|-------------------------|-----------------------|----------------------------|---------------------|--------------------------|-------------|
| 30 | 0.5 | 0.632121 | 0.637985 | 0.000326 | 0.391223 | 0.067972 | 0.639348 | 0.004721 | 0.639336 | 0.004621 | MLE |
| | 1 | 0.060587 | 0.061539 | 8.50E-06 | 0.032059 | 0.000975 | 0.063299 | 1.49E-04 | 0.063297 | 1.48E-04 | MLE |
| | 1.5 | 0.01227 | 0.012475 | 3.68E-07 | 0.006421 | 4.08E-05 | 0.01257 | 5.16E-06 | 0.012569 | 5.15E-06 | MLE |
| | 2 | 0.003899 | 0.003962 | 3.80E-08 | 0.002053 | 4.02E-06 | 0.004053 | 6.33E-07 | 0.004053 | 6.32E-07 | MLE |
| | 2.5 | 0.001599 | 0.00162 | 6.97E-09 | 0.000856 | 6.92E-07 | 0.001665 | 1.05E-07 | 0.001665 | 1.04E-07 | MLE |
| 60 | 0.5 | 0.632121 | 0.614304 | 0.000472 | 0.392887 | 0.062893 | 0.635537 | 0.002206 | 0.635534 | 0.002106 | MLE |
| | 1 | 0.060587 | 0.05781 | 1.09E-05 | 0.031457 | 0.000915 | 0.06179 | 6.62E-05 | 0.061789 | 6.61E-05 | MLE |
| | 1.5 | 0.01227 | 0.011698 | 4.62E-07 | 0.006302 | 3.82E-05 | 0.012515 | 2.62E-06 | 0.012515 | 2.61E-06 | MLE |
| | 2 | 0.003899 | 0.003714 | 4.77E-08 | 0.002002 | 3.84E-06 | 0.003961 | 2.62E-07 | 0.003961 | 2.61E-07 | MLE |
| | 2.5 | 0.001599 | 0.001524 | 7.92E-09 | 0.000821 | 6.53E-07 | 0.00163 | 4.42E-08 | 0.00163 | 4.41E-08 | MLE |
| 90 | 0.5 | 0.632121 | 0.605102 | 0.000829 | 0.391161 | 0.062047 | 0.63469 | 0.001488 | 0.634689 | 0.001388 | MLE |
| | 1 | 0.060587 | 0.056378 | 1.97E-05 | 0.031249 | 0.000901 | 0.060951 | 3.71E-05 | 0.060951 | 3.70E-05 | MLE |
| | 1.5 | 0.01227 | 0.011408 | 8.32E-07 | 0.006192 | 3.85E-05 | 0.012415 | 1.85E-06 | 0.012415 | 1.84E-06 | MLE |
| | 2 | 0.003899 | 0.003615 | 9.07E-08 | 0.002009 | 3.76E-06 | 0.003948 | 1.71E-07 | 0.003948 | 1.70E-07 | MLE |
| | 2.5 | 0.001599 | 0.001486 | 1.42E-08 | 0.000804 | 6.59E-07 | 0.001608 | 3.02E-08 | 0.001608 | 3.01E-08 | MLE |
| 120 | 0.5 | 0.632121 | 0.59974 | 0.001126 | 0.392628 | 0.06053 | 0.633521 | 0.001155 | 0.63352 | 0.001055 | MLE |
| | 1 | 0.060587 | 0.055582 | 2.67E-05 | 0.031136 | 0.000901 | 0.061123 | 3.04E-05 | 0.061123 | 3.034E-05 | MLE |
| | 1.5 | 0.01227 | 0.011241 | 1.13E-06 | 0.006228 | 3.79E-05 | 0.012371 | 1.28E-06 | 0.012371 | 1.27E-06 | MLE |
| | 2 | 0.003899 | 0.003565 | 1.18E-07 | 0.001983 | 3.81E-06 | 0.003934 | 1.21E-07 | 0.003934 | 1.20E-07 | MLE |
| | 2.5 | 0.001599 | 0.001463 | 1.97E-08 | 0.000815 | 6.78E-07 | 0.001609 | 2.02E-08 | 0.001609 | 2.01E-08 | MLE |
| 150 | 0.5 | 0.632121 | 0.595825 | 0.001375 | 0.39425 | 0.058786 | 0.634199 | 0.000887 | 0.634198 | 0.000877 | MLE |
| | 1 | 0.060587 | 0.055074 | 3.17E-05 | 0.031069 | 0.000897 | 0.060904 | 2.31E-05 | 0.060904 | 2.30E-05 | MLE |
| | 1.5 | 0.01227 | 0.011124 | 1.37E-06 | 0.006239 | 3.74E-05 | 0.012371 | 9.93E-07 | 0.012371 | 9.92E-07 | MLE |
| | 2 | 0.003899 | 0.003537 | 1.37E-07 | 0.001967 | 3.85E-06 | 0.003903 | 1.02E-07 | 0.003903 | 1.01E-07 | MLE |
| | 2.5 | 0.001599 | 0.001448 | 2.35E-08 | 0.000809 | 6.40E-07 | 0.001612 | 1.69E-08 | 0.001612 | 1.68E-08 | MLE |

Table (12): Reliability estimator when ($\lambda = 2, \theta = 1, c = 2$)

| <i>n</i> | <i>t_i</i> | <i>Real</i> | \hat{R}_{LS} | $MSE(\hat{R}_{LS})$ | \hat{R}_{LM} | $MSE(\hat{R}_{LM})$ | \hat{R}_{Bayes} | $MSE(\hat{R}_{Bayes})$ | \hat{R}_{Mle} | $MSE(\hat{R}_{Mle})$ | <i>Best</i> |
|----------|----------------------|-------------|----------------|---------------------|----------------|---------------------|-------------------|------------------------|-----------------|----------------------|-------------|
| 30 | 0.5 | 0.981684 | 0.219374 | 0.585654 | 0.820053 | 0.039134 | 0.981001 | 0.000168 | 0.979797 | 0.000188 | Bayes |
| | 1 | 0.632121 | 0.061782 | 3.26E-01 | 0.380058 | 0.083688 | 0.642127 | 4.21E-03 | 0.636145 | 4.13E-03 | MLE |
| | 1.5 | 0.35882 | 0.027426 | 1.10E-01 | 0.206 | 3.68E-02 | 0.372965 | 3.30E-03 | 0.36819 | 3.13E-03 | MLE |
| | 2 | 0.221199 | 0.015647 | 4.23E-02 | 0.122815 | 1.62E-02 | 0.231286 | 1.55E-03 | 0.22798 | 1.46E-03 | MLE |
| | 2.5 | 0.147856 | 0.010122 | 1.90E-02 | 0.080857 | 8.42E-03 | 0.154324 | 7.12E-04 | 0.152003 | 6.69E-04 | MLE |
| 60 | 0.5 | 0.981684 | 0.210307 | 0.597705 | 0.833267 | 0.03003 | 0.981535 | 8.14E-05 | 0.980934 | 8.60E-05 | Bayes |
| | 1 | 0.632121 | 0.057829 | 3.30E-01 | 0.386637 | 0.074135 | 0.639205 | 2.29E-03 | 0.636178 | 2.26E-03 | MLE |
| | 1.5 | 0.35882 | 0.026327 | 1.11E-01 | 0.196313 | 3.19E-02 | 0.364747 | 1.38E-03 | 0.362369 | 1.35E-03 | MLE |
| | 2 | 0.221199 | 0.014681 | 4.27E-02 | 0.123216 | 1.53E-02 | 0.226453 | 7.11E-04 | 0.224813 | 6.88E-04 | MLE |
| | 2.5 | 0.147856 | 0.009456 | 1.92E-02 | 0.080772 | 6.92E-03 | 0.1514 | 3.43E-04 | 0.150249 | 3.32E-04 | MLE |
| 90 | 0.5 | 0.981684 | 0.20695 | 0.602075 | 0.840121 | 0.025858 | 0.981096 | 5.44E-05 | 0.980685 | 5.68E-05 | Bayes |
| | 1 | 0.632121 | 0.055844 | 3.32E-01 | 0.39071 | 0.067486 | 0.637116 | 1.38E-03 | 0.635088 | 1.37E-03 | MLE |
| | 1.5 | 0.35882 | 0.02553 | 1.11E-01 | 0.200128 | 3.09E-02 | 0.363529 | 9.41E-04 | 0.361943 | 9.23E-04 | MLE |
| | 2 | 0.221199 | 0.014395 | 4.28E-02 | 0.119227 | 1.26E-02 | 0.224107 | 4.18E-04 | 0.22302 | 4.10E-04 | MLE |
| | 2.5 | 0.147856 | 0.009182 | 1.92E-02 | 0.077773 | 5.75E-03 | 0.149788 | 2.09E-04 | 0.149026 | 2.05E-04 | MLE |
| 120 | 0.5 | 0.981684 | 0.202511 | 0.608554 | 0.849082 | 0.022068 | 0.981749 | 4.09E-05 | 0.981448 | 4.19E-05 | Bayes |
| | 1 | 0.632121 | 0.055249 | 3.33E-01 | 0.390358 | 0.067724 | 0.636011 | 1.09E-03 | 0.634487 | 1.08E-03 | MLE |
| | 1.5 | 0.35882 | 0.024984 | 1.11E-01 | 0.200875 | 2.97E-02 | 0.361383 | 7.87E-04 | 0.360196 | 7.79E-04 | MLE |
| | 2 | 0.221199 | 0.014117 | 4.29E-02 | 0.119323 | 1.25E-02 | 0.223406 | 3.14E-04 | 0.222592 | 3.09E-04 | MLE |
| | 2.5 | 0.147856 | 0.009131 | 1.92E-02 | 0.076915 | 5.68E-03 | 0.149565 | 1.54E-04 | 0.148993 | 1.51E-04 | MLE |
| 150 | 0.5 | 0.981684 | 0.202846 | 0.607812 | 0.847306 | 0.021639 | 0.981088 | 3.46E-05 | 0.98084 | 3.56E-05 | MLE |
| | 1 | 0.632121 | 0.055208 | 3.33E-01 | 0.390436 | 0.065992 | 0.633482 | 9.09E-04 | 0.632262 | 9.07E-04 | MLE |
| | 1.5 | 0.35882 | 0.024724 | 1.12E-01 | 0.200325 | 2.83E-02 | 0.360309 | 5.42E-04 | 0.35936 | 5.38E-04 | MLE |
| | 2 | 0.221199 | 0.01404 | 4.29E-02 | 0.119387 | 1.31E-02 | 0.223626 | 2.90E-04 | 0.222973 | 2.86E-04 | MLE |
| | 2.5 | 0.147856 | 0.009007 | 1.93E-02 | 0.077084 | 5.50E-03 | 0.149243 | 1.28E-04 | 0.148786 | 1.26E-04 | MLE |

Table (13): Reliability estimator when ($\lambda = 3, \theta = 1, c = 2$)

| <i>n</i> | <i>t_i</i> | <i>Real</i> | \hat{R}_{LS} | $MSE(\hat{R}_{LS})$ | \hat{R}_{LM} | $MSE(\hat{R}_{LM})$ | \hat{R}_{Bayes} | $MSE(\hat{R}_{Bayes})$ | \hat{R}_{Mle} | $MSE(\hat{R}_{Mle})$ | <i>Best</i> |
|----------|----------------------|-------------|----------------|---------------------|----------------|---------------------|-------------------|------------------------|-----------------|----------------------|-------------|
| 30 | 0.5 | 0.999665 | 0.3425 | 0.433663 | 0.963761 | 0.002624 | 0.999362 | 9.98E-07 | 0.999338 | 1.07E-06 | Bayes |
| | 1 | 0.632121 | 0.050856 | 0.337921 | 0.393945 | 0.069609 | 0.645819 | 0.00473 | 0.64381 | 0.004682 | MLE |
| | 1.5 | 0.256433 | 0.015435 | 0.058085 | 0.140067 | 0.01665 | 0.264338 | 0.001749 | 0.26309 | 0.001718 | MLE |
| | 2 | 0.117503 | 0.006497 | 0.012323 | 0.063049 | 0.003624 | 0.122662 | 5.35E-04 | 0.122026 | 5.24E-04 | MLE |
| | 2.5 | 0.061995 | 0.003337 | 0.003441 | 0.033219 | 0.001234 | 0.064428 | 1.45E-04 | 0.064082 | 1.42E-04 | MLE |
| 60 | 0.5 | 0.999665 | 0.326167 | 0.454503 | 0.973132 | 0.001254 | 0.999549 | 2.22E-07 | 0.999539 | 2.32E-07 | MLE |
| | 1 | 0.632121 | 0.04839 | 0.340771 | 0.393219 | 0.065018 | 0.635863 | 0.00233 | 0.634849 | 0.002324 | MLE |
| | 1.5 | 0.256433 | 0.014704 | 0.058435 | 0.138464 | 0.015965 | 0.258944 | 0.000831 | 0.258329 | 0.000825 | MLE |
| | 2 | 0.117503 | 0.006199 | 0.012389 | 0.061459 | 0.003423 | 0.119852 | 2.29E-04 | 0.11954 | 2.27E-04 | MLE |
| | 2.5 | 0.061995 | 0.003175 | 0.00346 | 0.032537 | 0.000989 | 0.062767 | 6.03E-05 | 0.062599 | 5.98E-05 | Bayes |
| 90 | 0.5 | 0.999665 | 0.321625 | 0.460318 | 0.975374 | 0.000905 | 0.999575 | 1.26E-07 | 0.999569 | 1.30E-07 | Bayes |
| | 1 | 0.632121 | 0.047233 | 0.342111 | 0.395735 | 0.061219 | 0.635508 | 0.001379 | 0.63483 | 0.001375 | MLE |
| | 1.5 | 0.256433 | 0.014297 | 0.058631 | 0.138644 | 0.014803 | 0.258821 | 0.000606 | 0.258411 | 0.000602 | MLE |
| | 2 | 0.117503 | 0.006046 | 0.012423 | 0.062221 | 0.003512 | 0.119674 | 1.52E-04 | 0.119466 | 1.51E-04 | MLE |
| | 2.5 | 0.061995 | 0.003084 | 0.003471 | 0.03259 | 0.00096 | 0.062831 | 4.51E-05 | 0.062718 | 4.47E-05 | MLE |
| 120 | 0.5 | 0.999665 | 0.318412 | 0.464548 | 0.976907 | 0.000753 | 0.999605 | 8.72E-08 | 0.999601 | 8.93E-08 | Bayes |
| | 1 | 0.632121 | 0.0468 | 0.342613 | 0.395181 | 0.060306 | 0.632753 | 0.001096 | 0.632244 | 0.001095 | MLE |
| | 1.5 | 0.256433 | 0.014071 | 0.058741 | 0.140478 | 0.014454 | 0.257787 | 0.000392 | 0.25748 | 0.000391 | MLE |
| | 2 | 0.117503 | 0.005995 | 0.012434 | 0.061252 | 0.003325 | 0.11787 | 1.09E-04 | 0.117717 | 1.08E-04 | MLE |
| | 2.5 | 0.061995 | 0.003072 | 0.003472 | 0.03189 | 0.000955 | 0.062408 | 3.34E-05 | 0.062324 | 3.33E-05 | MLE |
| 150 | 0.5 | 0.999665 | 0.317061 | 0.466299 | 0.977513 | 0.000686 | 0.99961 | 6.27E-08 | 0.999606 | 6.40E-08 | Bayes |
| | 1 | 0.632121 | 0.046489 | 0.342975 | 0.393747 | 0.060178 | 0.63296 | 0.000953 | 0.632552 | 0.000952 | MLE |
| | 1.5 | 0.256433 | 0.013984 | 0.058782 | 0.13939 | 0.014325 | 0.258321 | 0.000345 | 0.258075 | 0.000344 | MLE |
| | 2 | 0.117503 | 0.005901 | 0.012455 | 0.0616 | 0.003267 | 0.118554 | 7.87E-05 | 0.118431 | 7.83E-05 | MLE |
| | 2.5 | 0.061995 | 0.003036 | 0.003476 | 0.031849 | 0.000945 | 0.062444 | 2.48E-05 | 0.062377 | 2.47E-05 | MLE |

Table (14): Reliability estimator when ($\lambda = 4, \theta = 1, c = 2$)

| n | t_i | <i>Real</i> | \widehat{R}_{LS} | $MSE(\widehat{R}_{LS})$ | \widehat{R}_{LM} | $MSE(\widehat{R}_{LM})$ | \widehat{R}_{Bayes} | $MSE(\widehat{R}_{Bayes})$ | \widehat{R}_{Mle} | $MSE(\widehat{R}_{Mle})$ | <i>Best</i> |
|-----|-------|-------------|--------------------|-------------------------|--------------------|-------------------------|-----------------------|----------------------------|---------------------|--------------------------|-------------|
| 30 | 0.5 | 1 | 0.424605 | 0.331931 | 0.998054 | 1.73E-05 | 0.999999 | 5.78E-11 | 0.999999 | 5.77E-11 | MLE |
| | 1 | 0.632121 | 0.033974 | 3.58E-01 | 0.395768 | 0.067242 | 0.637514 | 4.21E-03 | 0.637502 | 4.20E-03 | MLE |
| | 1.5 | 0.179245 | 0.006792 | 2.97E-02 | 0.096924 | 8.17E-03 | 0.18422 | 9.84E-04 | 0.184214 | 9.83E-04 | MLE |
| | 2 | 0.060587 | 0.002158 | 3.41E-03 | 0.032235 | 9.56E-04 | 0.062948 | 1.37E-04 | 0.062945 | 1.36E-04 | MLE |
| | 2.5 | 0.025275 | 0.000881 | 5.95E-04 | 0.013341 | 1.69E-04 | 0.026123 | 2.48E-05 | 0.026122 | 2.47E-05 | MLE |
| 60 | 0.5 | 1 | 0.403436 | 0.356286 | 0.998987 | 3.56E-06 | 1 | 1.43E-12 | 1 | 1.42E-12 | MLE |
| | 1 | 0.632121 | 0.031773 | 3.60E-01 | 0.396362 | 0.061497 | 0.636092 | 2.23E-03 | 0.636088 | 2.22E-03 | MLE |
| | 1.5 | 0.179245 | 0.006376 | 2.99E-02 | 0.095793 | 7.62E-03 | 0.181973 | 4.69E-04 | 0.181971 | 4.68E-04 | MLE |
| | 2 | 0.060587 | 0.002015 | 3.43E-03 | 0.031803 | 9.06E-04 | 0.061677 | 5.97E-05 | 0.061677 | 5.96E-05 | MLE |
| | 2.5 | 0.025275 | 0.000831 | 5.98E-04 | 0.012847 | 1.67E-04 | 0.025665 | 1.12E-05 | 0.025665 | 1.11E-05 | MLE |
| 90 | 0.5 | 1 | 0.395117 | 0.366189 | 0.999211 | 1.89E-06 | 1 | 5.85E-13 | 1 | 5.84E-13 | MLE |
| | 1 | 0.632121 | 0.031016 | 3.61E-01 | 0.394628 | 0.059996 | 0.631864 | 1.52E-03 | 0.631862 | 1.51E-03 | MLE |
| | 1.5 | 0.179245 | 0.00619 | 2.99E-02 | 0.096054 | 7.31E-03 | 0.180595 | 2.92E-04 | 0.180594 | 2.91E-04 | MLE |
| | 2 | 0.060587 | 0.001962 | 3.44E-03 | 0.031316 | 9.00E-04 | 0.061129 | 3.94E-05 | 0.061129 | 3.93E-05 | MLE |
| | 2.5 | 0.025275 | 0.000806 | 5.99E-04 | 0.012855 | 1.62E-04 | 0.025554 | 7.49E-06 | 0.025554 | 7.48E-06 | MLE |
| 120 | 0.5 | 1 | 0.391368 | 0.370615 | 0.999347 | 1.08E-06 | 1 | 2.02E-13 | 1 | 2.01E-13 | MLE |
| | 1 | 0.632121 | 0.030565 | 3.62E-01 | 0.393585 | 0.05972 | 0.633426 | 1.14E-03 | 0.633425 | 1.13E-03 | MLE |
| | 1.5 | 0.179245 | 0.006097 | 3.00E-02 | 0.095296 | 7.31E-03 | 0.180688 | 2.29E-04 | 0.180688 | 2.28E-04 | MLE |
| | 2 | 0.060587 | 0.001938 | 3.44E-03 | 0.031019 | 9.05E-04 | 0.061025 | 2.98E-05 | 0.061025 | 2.97E-05 | MLE |
| | 2.5 | 0.025275 | 0.000794 | 5.99E-04 | 0.012756 | 1.62E-04 | 0.025465 | 5.04E-06 | 0.025465 | 5.03E-06 | MLE |
| 150 | 0.5 | 1 | 0.388202 | 0.374476 | 0.999379 | 9.43E-07 | 1 | 9.06E-14 | 1 | 9.05E-14 | MLE |
| | 1 | 0.632121 | 0.030271 | 3.62E-01 | 0.394508 | 0.058926 | 0.633248 | 1.02E-03 | 0.633248 | 1.01E-03 | MLE |
| | 1.5 | 0.179245 | 0.00605 | 3.00E-02 | 0.09533 | 7.28E-03 | 0.179948 | 1.77E-04 | 0.179948 | 1.76E-04 | MLE |
| | 2 | 0.060587 | 0.001923 | 3.44E-03 | 0.030797 | 9.11E-04 | 0.06077 | 2.43E-05 | 0.06077 | 2.42E-05 | MLE |
| | 2.5 | 0.025275 | 0.000787 | 6.00E-04 | 0.012729 | 1.62E-04 | 0.025384 | 4.21E-06 | 0.025384 | 4.20E-06 | MLE |

Table (15): Reliability estimator when ($\lambda = 2, \theta = 1.5, c = 1$)

| n | t_i | <i>Real</i> | \widehat{R}_{LS} | $MSE(\widehat{R}_{LS})$ | \widehat{R}_{LM} | $MSE(\widehat{R}_{LM})$ | \widehat{R}_{Bayes} | $MSE(\widehat{R}_{Bayes})$ | \widehat{R}_{Mle} | $MSE(\widehat{R}_{Mle})$ | <i>Best</i> |
|-----|-------|-------------|--------------------|-------------------------|--------------------|-------------------------|-----------------------|----------------------------|---------------------|--------------------------|-------------|
| 30 | 0.5 | 0.999877 | 0.106974 | 0.79955 | 0.966255 | 2.62E-03 | 0.999734 | 2.49E-07 | 0.999717 | 2.78E-07 | Bayes |
| | 1 | 0.894601 | 0.028333 | 7.51E-01 | 0.63607 | 0.088195 | 0.896254 | 1.76E-03 | 0.894356 | 1.80E-03 | Bayes |
| | 1.5 | 0.632121 | 0.012506 | 3.84E-01 | 0.376482 | 8.43E-02 | 0.638787 | 4.15E-03 | 0.63577 | 4.12E-03 | MLE |
| | 2 | 0.430217 | 0.007151 | 1.79E-01 | 0.247794 | 5.00E-02 | 0.439324 | 3.67E-03 | 0.436653 | 3.61E-03 | MLE |
| | 2.5 | 0.302324 | 0.004573 | 8.87E-02 | 0.166311 | 2.81E-02 | 0.313733 | 2.52E-03 | 0.3116 | 2.46E-03 | MLE |
| 60 | 0.5 | 0.999877 | 0.099024 | 0.812697 | 0.975955 | 1.16E-03 | 0.99982 | 4.90E-08 | 0.999814 | 5.25E-08 | MLE |
| | 1 | 0.894601 | 0.024902 | 7.56E-01 | 0.65852 | 0.070488 | 0.895357 | 9.40E-04 | 0.894389 | 9.49E-04 | Bayes |
| | 1.5 | 0.632121 | 0.011352 | 3.85E-01 | 0.387738 | 7.46E-02 | 0.634883 | 2.33E-03 | 0.633364 | 2.32E-03 | MLE |
| | 2 | 0.430217 | 0.006459 | 1.80E-01 | 0.246909 | 4.28E-02 | 0.437454 | 1.83E-03 | 0.436113 | 1.80E-03 | MLE |
| | 2.5 | 0.302324 | 0.0041 | 8.89E-02 | 0.167438 | 2.47E-02 | 0.306001 | 1.16E-03 | 0.304949 | 1.15E-03 | MLE |
| 90 | 0.5 | 0.999877 | 0.09524 | 0.819205 | 0.978835 | 7.89E-04 | 0.999833 | 2.61E-08 | 0.999829 | 2.74E-08 | Bayes |
| | 1 | 0.894601 | 0.024439 | 7.57E-01 | 0.658297 | 0.067238 | 0.894618 | 6.05E-04 | 0.893966 | 6.09E-04 | Bayes |
| | 1.5 | 0.632121 | 0.010799 | 3.86E-01 | 0.393111 | 6.77E-02 | 0.635907 | 1.55E-03 | 0.634891 | 1.54E-03 | MLE |
| | 2 | 0.430217 | 0.006268 | 1.80E-01 | 0.243442 | 4.10E-02 | 0.432569 | 1.16E-03 | 0.43168 | 1.15E-03 | MLE |
| | 2.5 | 0.302324 | 0.003959 | 8.90E-02 | 0.164501 | 2.16E-02 | 0.305174 | 6.89E-04 | 0.304474 | 6.83E-04 | MLE |
| 120 | 0.5 | 0.999877 | 0.093022 | 0.823183 | 0.980041 | 6.90E-04 | 0.999841 | 1.93E-08 | 0.999839 | 2.01E-08 | Bayes |
| | 1 | 0.894601 | 0.02369 | 7.59E-01 | 0.66737 | 0.061493 | 0.895143 | 4.77E-04 | 0.894655 | 4.79E-04 | Bayes |
| | 1.5 | 0.632121 | 0.0109 | 3.86E-01 | 0.385422 | 6.80E-02 | 0.632434 | 1.09E-03 | 0.631671 | 1.08E-03 | MLE |
| | 2 | 0.430217 | 0.006077 | 1.80E-01 | 0.243484 | 3.98E-02 | 0.433474 | 8.53E-04 | 0.432805 | 8.47E-04 | MLE |
| | 2.5 | 0.302324 | 0.003906 | 8.91E-02 | 0.164959 | 2.22E-02 | 0.303867 | 5.00E-04 | 0.303343 | 4.97E-04 | MLE |
| 150 | 0.5 | 0.999877 | 0.090527 | 0.827515 | 0.982205 | 5.30E-04 | 0.99985 | 1.29E-08 | 0.999848 | 1.34E-08 | Bayes |
| | 1 | 0.894601 | 0.023862 | 7.58E-01 | 0.660535 | 0.062177 | 0.894207 | 3.97E-04 | 0.893814 | 3.99E-04 | Bayes |
| | 1.5 | 0.632121 | 0.010431 | 3.87E-01 | 0.395235 | 6.39E-02 | 0.633851 | 9.13E-04 | 0.633239 | 9.12E-04 | MLE |
| | 2 | 0.430217 | 0.005913 | 1.80E-01 | 0.247746 | 3.78E-02 | 0.432282 | 7.08E-04 | 0.431748 | 7.05E-04 | MLE |
| | 2.5 | 0.302324 | 0.003867 | 8.91E-02 | 0.163863 | 2.18E-02 | 0.30367 | 4.45E-04 | 0.303251 | 4.43E-04 | MLE |

Table (16): Reliability estimator when $(\lambda = 4, \theta = 1.5, c = 1)$

| n | t_i | Real | \hat{R}_{LS} | $MSE(\hat{R}_{LS})$ | \hat{R}_{LM} | $MSE(\hat{R}_{LM})$ | \hat{R}_{Bayes} | $MSE(\hat{R}_{Bayes})$ | \hat{R}_{Mle} | $MSE(\hat{R}_{Mle})$ | Best |
|-----|-------|----------|----------------|---------------------|----------------|---------------------|-------------------|------------------------|-----------------|----------------------|-------|
| 30 | 0.5 | 1 | 0.258415 | 0.550843 | 1 | 1.10E-18 | 1 | 0.00E+00 | 1 | 0.00E+00 | Bayes |
| | 1 | 0.99367 | 0.01853 | 9.51E-01 | 0.902607 | 0.012774 | 0.992236 | 4.52E-05 | 0.992381 | 4.35E-05 | MLE |
| | 1.5 | 0.632121 | 0.003701 | 3.95E-01 | 0.400364 | 6.52E-02 | 0.637574 | 4.78E-03 | 0.639066 | 4.80E-03 | Bayes |
| | 2 | 0.271237 | 0.001169 | 7.29E-02 | 0.154772 | 1.72E-02 | 0.280014 | 2.15E-03 | 0.280986 | 2.18E-03 | Bayes |
| | 2.5 | 0.121553 | 0.000483 | 1.47E-02 | 0.065382 | 3.78E-03 | 0.124492 | 5.29E-04 | 0.124972 | 5.36E-04 | Bayes |
| 60 | 0.5 | 1 | 0.243978 | 0.571992 | 1 | 1.93E-22 | 1 | 0.00E+00 | 1 | 0.00E+00 | Bayes |
| | 1 | 0.99367 | 0.017436 | 9.53E-01 | 0.910478 | 0.009371 | 0.992721 | 2.30E-05 | 0.992792 | 2.25E-05 | MLE |
| | 1.5 | 0.632121 | 0.003448 | 3.95E-01 | 0.397936 | 6.05E-02 | 0.636898 | 2.29E-03 | 0.637655 | 2.30E-03 | Bayes |
| | 2 | 0.271237 | 0.001094 | 7.30E-02 | 0.149405 | 1.63E-02 | 0.275024 | 9.92E-04 | 0.275506 | 9.99E-04 | Bayes |
| | 2.5 | 0.121553 | 0.000448 | 1.47E-02 | 0.064474 | 3.55E-03 | 0.123041 | 2.27E-04 | 0.123279 | 2.28E-04 | Bayes |
| 90 | 0.5 | 1 | 0.239059 | 0.579311 | 1 | 4.51E-24 | 1 | 0.00E+00 | 1 | 0.00E+00 | Bayes |
| | 1 | 0.99367 | 0.016931 | 9.54E-01 | 0.911626 | 0.00838 | 0.993117 | 1.36E-05 | 0.993163 | 1.35E-05 | MLE |
| | 1.5 | 0.632121 | 0.003363 | 3.95E-01 | 0.397866 | 5.90E-02 | 0.634376 | 1.51E-03 | 0.634883 | 1.51E-03 | Bayes |
| | 2 | 0.271237 | 0.001065 | 7.30E-02 | 0.149455 | 1.56E-02 | 0.273962 | 6.03E-04 | 0.274284 | 6.06E-04 | Bayes |
| | 2.5 | 0.121553 | 0.000437 | 1.47E-02 | 0.063641 | 3.51E-03 | 0.122698 | 1.46E-04 | 0.122857 | 1.47E-04 | Bayes |
| 120 | 0.5 | 1 | 0.23606 | 0.583805 | 1 | 5.24E-25 | 1 | 0.00E+00 | 1 | 0.00E+00 | Bayes |
| | 1 | 0.99367 | 0.016666 | 9.55E-01 | 0.915379 | 0.007321 | 0.993307 | 9.21E-06 | 0.993341 | 9.11E-06 | MLE |
| | 1.5 | 0.632121 | 0.003319 | 3.95E-01 | 0.394854 | 5.92E-02 | 0.634312 | 1.20E-03 | 0.634692 | 1.20E-03 | Bayes |
| | 2 | 0.271237 | 0.001048 | 7.30E-02 | 0.147985 | 1.59E-02 | 0.272184 | 4.66E-04 | 0.272424 | 4.67E-04 | Bayes |
| | 2.5 | 0.121553 | 0.00043 | 1.47E-02 | 0.063234 | 3.52E-03 | 0.122481 | 1.08E-04 | 0.1226 | 1.08E-04 | Bayes |
| 150 | 0.5 | 1 | 0.2337 | 0.587385 | 1 | 2.94E-23 | 1 | 0.00E+00 | 1 | 0.00E+00 | Bayes |
| | 1 | 0.99367 | 0.016497 | 9.55E-01 | 0.917001 | 0.006825 | 0.993434 | 6.58E-06 | 0.993461 | 6.52E-06 | MLE |
| | 1.5 | 0.632121 | 0.003271 | 3.95E-01 | 0.398861 | 5.68E-02 | 0.63344 | 9.07E-04 | 0.633745 | 9.07E-04 | Bayes |
| | 2 | 0.271237 | 0.001042 | 7.30E-02 | 0.147334 | 1.58E-02 | 0.271978 | 3.55E-04 | 0.27217 | 3.55E-04 | Bayes |
| | 2.5 | 0.121553 | 0.000425 | 1.47E-02 | 0.063133 | 3.50E-03 | 0.122636 | 8.59E-05 | 0.122732 | 8.63E-05 | Bayes |

Conclusion

1. The best estimator for scale parameter is found by maximum likelihood first as explained by table (1)(2)(3)(4).and the second method is Bayes one ,and the third one is Least square method ,this is due to the important properties of ML Estimator
2. for certain chosen set values $(\lambda = 4, \theta = 0.5, c = 2)$, we find $[MSE(\hat{\theta})]$ using lest square is the smallest.
3. The best estimator of reliability function is also by maximum likelihood method since $\hat{\theta}_{ML}$ is the best estimator ,so this results Reflect on \hat{R}_{ML} .

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